13.5

Calculating Gradients

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient, together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$ **2.** $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$

3. $g(x, y) = xy^2, \quad (2, -1)$ **4.** $g(x, y) = \dfrac{x^2}{2} - \dfrac{y^2}{2}, \quad (\sqrt{2}, 1)$

Sol'n:

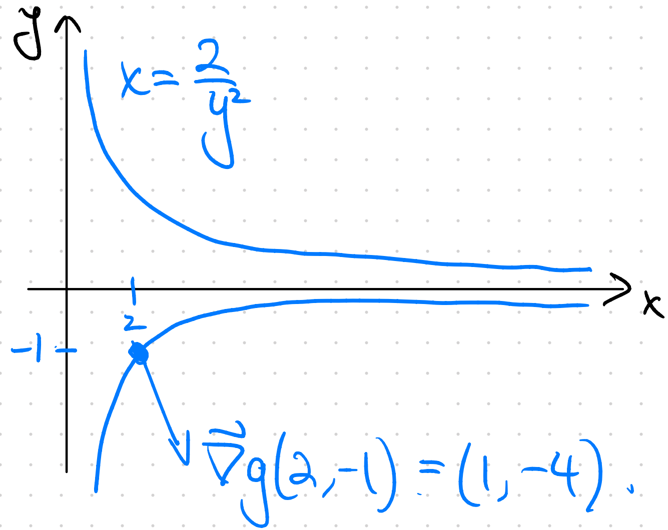
$$\left.\frac{\partial g}{\partial x}\right|_{(2,-1)} = \left. y^2 \right|_{(2,-1)} = 1, \quad \left.\frac{\partial g}{\partial y}\right|_{(2,-1)} = \left. 2xy \right|_{(2,-1)} = -4$$

So $\vec{\nabla} g(2,-1) = (1,-4)$.

$$g(2,-1) = 2(-1)^2 = 2.$$

So level curve that passes through the point is

$$xy^2 = 2 \Rightarrow x = \frac{2}{y^2}$$



13.5

Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at P_0 in the direction of $\mathbf{v}$.

11. $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$

12. $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

13. $g(x, y) = \dfrac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{v} = 12\mathbf{i} + 5\mathbf{j}$

Sol'n: We first need to take $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

$\|\vec{v}\| = \sqrt{12^2 + 5^2} = 13$ So $\vec{u} = \frac{1}{13}(12, 5)$.

Then $D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \left.\frac{(1)(xy+2) - (x-y)(y)}{(xy+2)^2}\right|_{(1,-1)} = \left.\frac{xy+2-xy+y^2}{(xy+2)^2}\right|_{(1,-1)}$$

$$= \frac{2+(-1)^2}{((1)(-1)+2)^2} = 3$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,-1)} = \left.\frac{(-1)(xy+2)-(x-y)(x)}{(xy+2)^2}\right|_{(1,-1)} = \left.\frac{-xy-2-x^2+xy}{(xy+2)^2}\right|_{(1,-1)}$$

$$= \frac{-2-(1)^2}{((1)(-1)+2)^2} = -3$$

So $D_{\vec{u}}f(P_0) = \vec{\nabla}f(P_0)\cdot\vec{u} = (3,-3)\cdot\frac{1}{13}(12,5) = \boxed{\frac{21}{13}}$

13.5 Theory and Examples

29. Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}}f(1,-1)$ for which

a. $D_{\mathbf{u}}f(1,-1)$ is largest

b. $D_{\mathbf{u}}f(1,-1)$ is smallest

c. $D_{\mathbf{u}}f(1,-1) = 0$

d. $D_{\mathbf{u}}f(1,-1) = 4$

e. $D_{\mathbf{u}}f(1,-1) = -3$

Sol'n: a) As discussed in lecture $-\|\vec{\nabla}f(\vec{a})\| \leq D_{\vec{u}}f(\vec{a}) \leq \|\vec{\nabla}f(\vec{a})\|$ with equality when $u = \mp \dfrac{\vec{\nabla}f(\vec{a})}{\|\vec{\nabla}f(\vec{a})\|}$ resp. for u unit vector.

Compute $\vec{\nabla}f(1,-1) = \left[\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} \quad \left.\frac{\partial f}{\partial y}\right|_{(1,-1)} \right]$

$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \left. 2x - y \right|_{(1,-1)} = 2-(-1) = 3$, $\left.\frac{\partial f}{\partial y}\right|_{(1,-1)} = \left. -x+2y-1 \right|_{(1,-1)} = -4.$

So $\vec{\nabla}f(1,-1) = [3 \ -4]$.

So $\|\vec{\nabla}f(1,-1)\| = \sqrt{3^2+(-4)^2} = 5$

So $D_{\vec{u}} f(1,-1)$ is largest when $\boxed{\vec{u} = \left(\frac{3}{5}, \frac{-4}{5}\right)}$

b) $D_{\vec{u}} f(1,-1)$ is smallest when $\boxed{\vec{u} = \left(\frac{-3}{5}, \frac{4}{5}\right)}$

c) For $\vec{u} = (u_1, u_2)$ unit vector,

$$D_{\vec{u}} f(1,-1) = \vec{\nabla} f(1,-1) \cdot \vec{u} = (3,-4) \cdot (u_1, u_2) = 3u_1 - 4u_2$$

$0 = 3u_1 - 4u_2$

So can take $u_1 = \frac{4}{5}$, $u_2 = \frac{3}{5}$. $\Rightarrow$ $\vec{u} = \left(\frac{4}{5}, \frac{3}{5}\right)$

$u_1 = \frac{-4}{5}$, $u_2 = \frac{-3}{5}$. $\Rightarrow$ $\vec{u} = \left(\frac{-4}{5}, \frac{-3}{5}\right)$

d) $4 = 3u_1 - 4u_2$.

$3u_1 = 4 + 4u_2$

$u_1 = \frac{4}{3} + \frac{4}{3} u_2$.

$1 = \sqrt{u_1^2 + u_2^2} \Rightarrow 1 = u_1^2 + u_2^2 \Rightarrow 1 = \left(\frac{4}{3} + \frac{4}{3} u_2\right)^2 + u_2^2$

$\Rightarrow \frac{25u_2^2}{9} + \frac{32u_2}{9} + \frac{7}{9} = 0. \Rightarrow u_2 = -1, \frac{-7}{25}.$

When $u_2=-1$, we have $u_1=0$. $\Rightarrow \boxed{\vec{u}=(0,-1)}$

$u_2=\frac{-7}{25}$, we have $4=3u_1-4\left(\frac{-7}{25}\right) \Rightarrow 3u_1=\frac{72}{25}$

$\Rightarrow u_1=\frac{24}{25}$.

$\Rightarrow \boxed{\vec{u}=\left(\frac{24}{25},\frac{-7}{25}\right)}$

e) $-3=3u_1-4u_2$.

$4u_2=3u_1+3$.

$u_2=\frac{3}{4}u_1+\frac{3}{4}$.

$1=u_1^2+u_2^2 \Rightarrow 1=u_1^2+\left(\frac{3}{4}u_1+\frac{3}{4}\right)^2=u_1^2+\frac{9u_1^2}{16}+\frac{9u_1}{8}+\frac{9}{16}$

$=\frac{25u_1^2}{16}+\frac{9u_1}{8}+\frac{9}{16}$

$\Rightarrow \frac{25u_1^2}{16}+\frac{9u_1}{8}-\frac{7}{16}=0$

$\Rightarrow u_1=-1, \frac{7}{25}$.

When $u_1=-1$, $u_2=0$ $\Rightarrow \boxed{\vec{u}=(-1,0)}$

$u_1=\frac{7}{25}$, $u_2=\frac{3}{4}\left(\frac{7}{25}\right)+\frac{3}{4}=\frac{24}{25}$ $\Rightarrow \boxed{\vec{u}=\left(\frac{7}{25},\frac{24}{25}\right)}$

13.6

23. By about how much will

$$g(x, y, z) = x + x\cos z - y\sin z + y$$

change if the point $P(x, y, z)$ moves from $P_0(2, -1, 0)$ a distance of $ds = 0.2$ unit toward the point $P_1(0, 1, 2)$?

Sol'n: $dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(P_0)\, dx_i$

$\frac{\partial g}{\partial x} = 1 + \cos z \qquad \left.\frac{\partial g}{\partial x}\right|_{P_0} = 1 + \cos(0) = 2.$

$\frac{\partial g}{\partial y} = -\sin z + 1 \qquad \left.\frac{\partial g}{\partial y}\right|_{P_0} = -\sin(0) + 1 = 1$

$\frac{\partial g}{\partial z} = -x\sin z - y\cos z. \qquad \left.\frac{\partial g}{\partial z}\right|_{P_0} = -2\sin(0) - (-1)\cos(0) = 1.$

$P - P_0 = (0, 1, 2) - (2, -1, 0) = (-2, 2, 2).$

$dx = -2ds = -2(0.2) \qquad dz = 2ds = 2(0.2).$

$dy = 2ds = 2(0.2)$

So $\Delta g \approx dg = 2(-2)(0.2) + 1(2)(0.2) + 1(2)(0.2) = \boxed{0}$

13.6

Linearizations for Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 41–46 at the given points.

45. $f(x, y, z) = e^x + \cos(y + z)$ at

a. $(0, 0, 0)$ **b.** $\left(0, \frac{\pi}{2}, 0\right)$ **c.** $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$

Sol'n: Linearization: $L(\vec{w}) = f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{w} - \vec{a})$.

$$\vec{\nabla} f = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right] = \left[e^x \quad -\sin(y+z) \quad -\sin(y+z)\right].$$

a) $f(\vec{a}) = f(0,0,0) = e^0 + \cos(0+0) = 1 + 1 = 2.$

$$\vec{\nabla} f(0,0,0) = \left[e^x \quad -\sin(y+z) \quad -\sin(y+z)\right]\Big|_{(0,0,0)}$$

$$= \left[e^0 \quad -\sin(0+0) \quad -\sin(0+0)\right]$$

$$= \left[1 \quad 0 \quad 0\right]$$

$\vec{w} = (x, y, z).$

So $L(\vec{w}) = 2 + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{2+x}$

b) $f(\vec{b}) = f(0, \frac{\pi}{2}, 0) = e^0 + \cos(\frac{\pi}{2} + 0) = 1 + \cos(\frac{\pi}{2}) = 1.$

$\vec{\nabla} f(0, \frac{\pi}{2}, 0) = \begin{bmatrix} e^0 & -\sin(\frac{\pi}{2}+0) & -\sin(\frac{\pi}{2}+0) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}.$

$\vec{\nabla} f(0, \frac{\pi}{2}, 0) \cdot (x, y - \frac{\pi}{2}, z) = x - (y - \frac{\pi}{2}) - z = x - y - z + \frac{\pi}{2}$

So $L(\vec{w}) = \boxed{x - y - z + \frac{\pi}{2} + 1}$

c) $f(\vec{c}) = f(0, \frac{\pi}{4}, \frac{\pi}{4}) = e^0 + \cos(\frac{\pi}{4} + \frac{\pi}{4}) = 1.$

$\vec{\nabla} f(0, \frac{\pi}{4}, \frac{\pi}{4}) = \begin{bmatrix} e^0 & -\sin(\frac{\pi}{4}+\frac{\pi}{4}) & -\sin(\frac{\pi}{4}+\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}.$

$\vec{\nabla} f(0, \frac{\pi}{4}, \frac{\pi}{4}) \cdot (x, y - \frac{\pi}{4}, z - \frac{\pi}{4}) = x + \frac{\pi}{4} - y + \frac{\pi}{4} - z.$

So $L(\vec{w}) = \boxed{x - y - z + \frac{\pi}{2} + 1}$